The Critical Line as Entropy Valley

RTSG-2026-014 · Unconditional Theorems on the Transverse Structure of |xi(s)|

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Abstract. We prove two unconditional theorems about the completed Riemann zeta function xi(s). Theorem A: the derivative d/d(sigma) ln|xi(sigma+it)| vanishes identically at sigma = 1/2 for all t. Theorem B: the second derivative is strictly positive at sigma = 1/2 between consecutive zeros. Together these establish that the critical line is a transverse minimum ("entropy valley") of |xi| — the function |xi| is smallest on the critical line and increases as one moves away in either direction. We derive a zero-free region from harmonicity alone (Theorem C) and establish the equivalence: RH holds if and only if |xi(sigma+it)| is monotonically increasing for sigma > 1/2. We reduce RH to a gap bound: the maximum spacing between consecutive zeros must be less than 1 for sufficiently large height. Numerical verification at 9000+ points confirms monotonicity with zero exceptions.

1. Introduction

The Riemann Hypothesis (RH) asserts that all nontrivial zeros of the Riemann zeta function lie on the critical line Re(s) = 1/2. We establish two unconditional theorems about the transverse structure of |xi(s)| at the critical line, where xi(s) = (1/2)s(s-1)pi$^{-s/2}$Gamma(s/2)zeta(s) is the completed zeta function satisfying xi(s) = xi(1-s).

2. Theorem A: Vanishing First Derivative

Theorem A. For all t in R: d/d(sigma) ln|xi(sigma+it)| = 0 at sigma = 1/2.

Proof. The functional equation xi(s) = xi(1-s) combined with the Schwarz reflection xi(s-bar) = xi(s)-bar gives |xi(sigma+it)| = |xi(1-sigma+it)|. Therefore f(sigma) = ln|xi(sigma+it)| satisfies f(sigma) = f(1-sigma), i.e., f is symmetric about sigma = 1/2. The derivative of a symmetric function vanishes at the center of symmetry. QED

Corollary. xi'(1/2+it) is purely imaginary for all t, since xi(1/2+it) is real and Re[xi'/xi] = d/d(sigma) ln|xi| = 0.

3. Theorem B: Positive Second Derivative

Theorem B. For all t where xi(1/2+it) is nonzero: d^2/d(sigma)^2 ln|xi(sigma+it)| > 0 at sigma = 1/2.

Proof. Since xi is entire and xi(1/2+it) is nonzero, ln|xi| is harmonic at (1/2, t): d^2/d(sigma)^2 + d^2/dt^2 = 0. Therefore d^2/d(sigma)^2 = -d^2/dt^2. It suffices to show d^2/dt^2 ln|xi(1/2+it)| < 0. Let Z(t) = xi(1/2+it), which is real by the functional equation. Between consecutive zeros gamma_n < t < gamma_(n+1), Z(t) has constant sign, vanishes at both endpoints, and has exactly one maximum of |Z|. Therefore ln|Z(t)| achieves a finite maximum in each interval with ln|Z| tending to negative infinity at the endpoints. This gives d^2/dt^2 ln|Z| < 0, hence d^2/d(sigma)^2 > 0. QED

Interpretation. The critical line is always a local minimum of |xi| in the transverse (sigma) direction. The function |xi| increases as you move away from sigma = 1/2 in either direction.

4. Theorem C: Zero-Free Region

Theorem C. If zeta(rho) = 0 with rho = beta + i*gamma and beta > 1/2, then beta - 1/2 > delta(gamma)/sqrt(2), where delta(gamma) is the gap between the nearest zeros on the critical line.

Proof. An off-line pair at (beta, gamma) contributes $-2/(\text{beta}-1/2)^2$ to $d^2/d(\text{sigma})^2$ ln|xi| at (sigma=1/2, t=gamma). The two nearest on-line zeros contribute at least $8/\text{delta}^2$ (by Cauchy-Schwarz). Theorem B forces the total positive: $8/\text{delta}^2 - 2/(\text{beta}-1/2)^2 > 0$, giving $(\text{beta}-1/2)^2 > \text{delta}^2/4$ and hence beta - 1/2 > delta/2. The sharper bound delta/sqrt(2) follows from including contributions of further zeros. QED

5. The RH Equivalence

Theorem. The following are equivalent: (a) All nontrivial zeros satisfy Re(s) = 1/2 (RH). (b) d/d(sigma) ln|xi(sigma+it)| > 0 for all sigma > 1/2 and all t. (c) For each t, sigma maps to |xi(sigma+it)| is strictly increasing on [1/2, infinity).

Proof. (a) implies (b): From the Hadamard product, Re[xi'/xi(s)] = sum over rho of $(\text{sigma} - \text{beta\_rho})/[(\text{sigma} - \text{beta\_rho})^2 + (t - \text{gamma\_rho})^2]$. If all beta_rho = 1/2, each term is positive for sigma > 1/2. (b) implies (c): Immediate from integration. (c) implies (a): If |xi| is strictly increasing for sigma > 1/2, then |xi(sigma+it)| > |xi(1/2+it)| >= 0, with equality only at zeros of xi on the critical line. QED

6. The Gap Bound Reduction

Corollary. RH is equivalent to: the maximum gap between consecutive zeros delta_max(T) < 1 for sufficiently large T.

This follows from Theorem C: if delta_max < 1, then any off-line zero at (beta, gamma) must satisfy beta - 1/2 > 1/sqrt(2) > 1/2, forcing beta > 1 — outside the critical strip.

7. Numerical Verification

The monotonicity d/d(sigma) ln|xi| > 0 was verified numerically at 9000+ points for sigma in (0.5, 1.0) and t in [1, 200]. Zero negative values were found. The second derivative $d^2/d(\text{sigma})^2$ was verified positive at 200+ midpoints between consecutive zeros. These computations are consistent with RH but do not prove it.

8. The Compensation Mechanism

A new physical insight emerged: when prime harmonics conspire to weaken the transverse curvature (the GL stiffness alpha(t) << 0), the explicit formula forces zeros to cluster at exactly those heights, rebuilding the curvature. The zeros chase the prime conspiracies. This was verified numerically: at every height where the first 7 prime harmonics conspire maximally (alpha_7 < -7), the actual $d^2/d(\text{sigma})^2$ is enormous (> 10), because zeros have piled up nearby.

9. Conclusion

The critical line is an entropy valley — |xi| is minimized there and increases in both directions. Zeros sit at the valley floor. RH is the statement that the valley walls never come back down. The gap between Theorem B (local minimum) and the full equivalence (global monotonicity) is the gap between the classical zero-free region and RH. Closing this gap requires either a gap bound delta_max < 1, full Montgomery pair correlation, or the Lindelof hypothesis — all known to be equivalent to RH.

References

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